

ON ZAMOLODCHIKOV'S PERIODICITY CONJECTURE FOR Y-SYSTEMS

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A . I prove Zamolodchikov's periodicity conjecture for type A with both ranks arbitrary.

I

Following Alexei Zamolodchikov [1], one can associate to any pair of indecomposable Cartan matrices of finite type the so called Y-system of algebraic equations, which reads

$$(1) \quad Y_{i,j+1,k} Y_{i,j-1,k} = \frac{\prod_{i' \neq i} (1 + Y_{i'jk})^{-a_{ii'}}}{\prod_{k' \neq k} (1 + 1/Y_{ijk'})^{-a'_{kk'}}}$$

$$i, j, k \in \mathbb{Z} \quad 1 \leq i \leq r \quad 1 \leq k \leq r',$$

where $(a_{ii'})$ and $(a'_{kk'})$ are those Cartan matrices, and r and r' are their ranks. Then the periodicity conjecture asserts that all solutions to this system are periodic in j , with period equal to twice the sum of the respective dual Coxeter numbers.

Until now this conjecture has only been proved in the case when one of the ranks equals 1 [2, 3, 4]. In this paper, we will take the next logical step and prove the case when the two Cartan matrices are of type A of arbitrary ranks r and r' .

Recall that Cartan matrices of type A are tridiagonal, with twos on the diagonal and minus ones on the sub- and sup-diagonals. Thus, for $1 < i < r$ and $1 < k < r'$ we have

$$Y_{i,j+1,k} Y_{i,j-1,k} = \frac{(1 + Y_{i+1,jk})(1 + Y_{i-1,jk})}{(1 + 1/Y_{ijk+1})(1 + 1/Y_{ijk-1})},$$

while at the boundaries one or two factors in the right hand side are absent—for instance,

$$Y_{1,j+1,k} Y_{1,j-1,k} = \frac{1 + Y_{2,jk}}{(1 + 1/Y_{1jk+1})(1 + 1/Y_{1jk-1})}$$

$$Y_{1,j+1,1} Y_{1,j-1,1} = \frac{1 + Y_{2,j1}}{1 + 1/Y_{1j2}}.$$

However, they can be added back if we introduce fictitious boundary variables and set them equal zero or infinity, as appropriate: $Y_{0jk} = Y_{r+1jk} = 0$, $Y_{ij0} = Y_{ijr'+1} = \infty$. Note that Y_{0j0} , $Y_{0jr'+1}$, Y_{r+1j0} and $Y_{r+1jr'+1}$ are thus ill-defined, but they never appear in the right hand side anyway.

Note also that our system consists of two completely decoupled identical subsystems—one involving variables Y_{ijk} with $i + j + k$ even, and the other with $i + j + k$ odd. We will, therefore, simply discard the second subsystem and assume that the Y_{ijk} are only defined for $i + j + k$ even.

Finally, recall that the dual Coxeter number for type A equals rank plus one. Thus, we have to prove the following.

Theorem (Zamolodchikov's conjecture, case AA). *All regular solutions (that is, solutions avoiding 0, -1 and ∞ everywhere except the boundaries) to the Y -system*

$$(2) \quad Y_{ij+1k} Y_{ij-1k} = \frac{(1 + Y_{i+1jk})(1 + Y_{i-1jk})}{(1 + 1/Y_{ijk+1})(1 + 1/Y_{ijk-1})}$$

$$1 \leq i \leq r \quad 1 \leq k \leq r' \quad i + j + k \text{ odd}$$

with the 'free boundary conditions'

$$(3) \quad Y_{0jk} = Y_{r+1jk} = 0 \quad Y_{ij0} = Y_{ijr'+1} = \infty$$

are $2(r + r' + 2)$ -periodic in j :

$$(4) \quad Y_{i,j+2(r+r'+2),k} = Y_{ijk} .$$

We prove this by producing a manifestly periodic formula for the general solution. This formula involves $r+r'+2$ arbitrary points of the r -dimensional projective space,

$$x_n \in \mathbb{CP}^r \quad x_{n+r+r'+2} = x_n ,$$

and reads

$$(5) \quad Y_{ijk} = -X_{\underbrace{x_{a+1} + \dots + x_{b-1}}_{i-1} + \underbrace{x_{c+1} + \dots + x_{d-1}}_{r-i}}(x_a, x_b, x_c, x_d) ,$$

where

$$a = \frac{-i - j - k}{2} \quad b = a + i = \frac{i - j - k}{2}$$

$$c = b + k = \frac{i - j + k}{2} \quad d = c + r + 1 - i = \frac{-i - j + k}{2} + r + 1 ,$$

and X is a certain $(r+3)$ -point projective invariant to be defined in Section 1.

As you can see, periodicity is indeed manifest, and so we shall be done once we check that a) the proposed solution does actually solve the Y -system (2), and b) that it is indeed a general solution. This is done in Sections 2 and 3.

I wish to thank L. Faddeev, I. Krichever, I. Loris, A. Szenes and especially A. Alekseev for helpful discussions.

1. C -

In this section we recall some textbook facts about projective geometry and define the projective invariant used in formula (5).

Recall that the projective space \mathbb{CP}^r is the set of all lines through the origin of \mathbb{C}^{r+1} , or in other words, it is the quotient of $\mathbb{C}^{r+1} \setminus \{0\}$ by the equivalence relation that two vectors are equivalent iff they are collinear. For a vector $\vec{x} \in \mathbb{C}^{r+1} \setminus \{0\}$ we denote the corresponding element (point) of \mathbb{CP}^r by $[\vec{x}]$ or simply x , and say that the former is a representative vector of the latter. Points are said to be projectively independent if their representative vectors are linearly independent.

The projective span $x_1 + \dots + x_N$ of two or more points $x_n \in \mathbb{CP}^r$ is the set of all points whose representative vectors lie in the linear span of the vectors \vec{x}_n . Projective spans of r projectively independent points are called hyperplanes.

To any invertible linear transformation T of \mathbb{C}^{r+1} we associate the invertible map t of \mathbb{CP}^r onto itself defined by $t([\vec{x}]) = [T\vec{x}]$. Such maps are called projective transformations, and they map projective spans to projective spans: $t(x_1 + \dots + x_N) = t(x_1) + \dots + t(x_N)$.

We say $r+2$ points of \mathbb{CP}^r to be in general position, or to form a projective frame, if no $r+1$ of them lie in the same hyperplane, or in other words, if no $r+1$ of their representative vectors are linearly dependent. Such frames are akin to vector bases in that for any two of them there is a unique projective transformation taking one to the other.

For instance, for $r = 1$ any three distinct points form a projective frame. Thus, they can be taken into any other three by a unique projective transformation, but this is not the case for four points. Four points in \mathbb{CP}^1 (of which at least three are distinct) have a numeric projective invariant, which is called the cross-ratio, and is defined by

$$X(x_1, x_2, x_3, x_4) = \frac{\vec{x}_1 \wedge \vec{x}_2}{\vec{x}_1 \wedge \vec{x}_4} \frac{\vec{x}_3 \wedge \vec{x}_4}{\vec{x}_3 \wedge \vec{x}_2}.$$

Note that division of bivectors makes proper sense here because they all are multiples of one and the same bivector. Explicitly, if x_n^l ($l = 1, 2$) are coordinates of vectors \vec{x}_n relative to some basis $\vec{e}_1, \vec{e}_2 \in \mathbb{C}^2$, and $|x_m x_n|$ denotes the determinant of the corresponding coordinate matrix,

$$|\vec{x}_m \vec{x}_n| = \det \begin{pmatrix} x_m^1 & x_n^1 \\ x_m^2 & x_n^2 \end{pmatrix}$$

then $\vec{x}_m \wedge \vec{x}_n = |\vec{x}_m \vec{x}_n| \vec{e}_1 \wedge \vec{e}_2$, and hence

$$X(x_1, x_2, x_3, x_4) = \frac{|\vec{x}_1 \vec{x}_2| |\vec{x}_3 \vec{x}_4|}{|\vec{x}_1 \vec{x}_4| |\vec{x}_3 \vec{x}_2|} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)},$$

where $z_n = x_n^1/x_n^2 \in \mathbb{C} \cup \{\infty\}$.

The last expression makes it particularly easy to check how the cross-ratio changes under permutations of the four points involved. It turns out

that 24 such permutations yield only 6 different values of the cross-ratio: if one of them equals ω , then the other five are $1/\omega$, $1 - \omega$, $1/(1 - \omega)$, $1 - 1/\omega$ and $1/(1 - 1/\omega)$. In particular, we have

$$X(x_1, x_3, x_2, x_4) = 1 - X(x_1, x_2, x_3, x_4)$$

$$X(x_1, x_2, x_4, x_3) = \frac{1}{1 - 1/X(x_1, x_2, x_3, x_4)},$$

which suitably explains why the Y-system involves Y , $1 + Y$ and $1/(1 + 1/Y)$ at the same time.

Back to the general case, we are now ready to define the invariant used in the general solution formula (5). Consider $r + 3$ points $x_1, \dots, x_{r+3} \in \mathbb{CP}^r$ such that any three of the first four plus the remaining $r - 1$ of them are in general position. Let V be the subspace of \mathbb{C}^{r+1} spanned by the representative vectors of those last $r - 1$ points x_5, \dots, x_{r+3} , and π the canonical projection $\mathbb{C}^{r+1} \rightarrow \mathbb{C}^{r+1}/V \simeq \mathbb{C}^2$. Then the multi-dimensional cross-ratio is defined by

$$(6) \quad X_{x_5 + \dots + x_{r+3}}(x_1, x_2, x_3, x_4) = X(x'_1, x'_2, x'_3, x'_4),$$

where $x'_n = [\pi(\vec{x}_n)] \in \mathbb{CP}^1$.

Note two things. First, that this generalized cross-ratio obviously has the same behaviour under permutation of the points x_1, \dots, x_4 as did the original cross-ratio. In particular, we again have

$$(7) \quad X_{x_5 + \dots + x_{r+3}}(x_1, x_3, x_2, x_4) = 1 - X_{x_5 + \dots + x_{r+3}}(x_1, x_2, x_3, x_4)$$

$$X_{x_5 + \dots + x_{r+3}}(x_1, x_2, x_4, x_3) = \frac{1}{1 - 1/X_{x_5 + \dots + x_{r+3}}(x_1, x_2, x_3, x_4)}.$$

Second, that, in terms of wedge products and determinants, the above definition becomes

$$(8) \quad X_{x_5 + \dots + x_{r+3}}(x_1, x_2, x_3, x_4) = \frac{\vec{x}_1 \wedge \vec{x}_2 \wedge \vec{x}_5 \wedge \dots \wedge \vec{x}_{r+3}}{\vec{x}_1 \wedge \vec{x}_4 \wedge \vec{x}_5 \wedge \dots \wedge \vec{x}_{r+3}}$$

$$\times \frac{\vec{x}_3 \wedge \vec{x}_4 \wedge \vec{x}_5 \wedge \dots \wedge \vec{x}_{r+3}}{\vec{x}_3 \wedge \vec{x}_2 \wedge \vec{x}_5 \wedge \dots \wedge \vec{x}_{r+3}} = \frac{|\vec{x}_1 \vec{x}_2 \vec{x}_5 \dots \vec{x}_{r+3}| |\vec{x}_3 \vec{x}_4 \vec{x}_5 \dots \vec{x}_{r+3}|}{|\vec{x}_1 \vec{x}_4 \vec{x}_5 \dots \vec{x}_{r+3}| |\vec{x}_3 \vec{x}_2 \vec{x}_5 \dots \vec{x}_{r+3}|}.$$

It is the latter expression that we mostly use in what follows.

2. P , (a)

Here we check that the Y 's given by formula (5) do indeed satisfy the Y-system (2).

First, we use eq. (8) to restate that formula in a fully explicit form. This time it involves $r + r' + 2$ arbitrary $(r + 1)$ -vectors

$$\vec{x}_n \in \mathbb{C}^{r+1} \quad \vec{x}_{n+r+r'+2} = \vec{x}_n,$$

and reads

$$(9) \quad Y_{ijk} = \frac{|\vec{x}_a \dots \vec{x}_b \vec{x}_{c+1} \dots \vec{x}_{d-1}| |\vec{x}_{a+1} \dots \vec{x}_{b-1} \vec{x}_c \dots \vec{x}_d|}{|\vec{x}_a \dots \vec{x}_{b-1} \vec{x}_{c+1} \dots \vec{x}_d| |\vec{x}_{a+1} \dots \vec{x}_b \vec{x}_c \dots \vec{x}_{d-1}|},$$

where, as before,

$$\begin{aligned} a &= \frac{-i-j-k}{2} & b &= a+i = \frac{i-j-k}{2} \\ c &= b+k = \frac{i-j+k}{2} & d &= c+r+1-i = \frac{-i-j+k}{2} + r+1. \end{aligned}$$

Note that the minus before the right hand side is gone because the order of columns here is different from that in the original definition (8). Note also that the inequalities $0 < i < r+1$ and $0 < k < r'+1$ are exactly equivalent to $a < b < c < d < a+r+r'+2$. This guarantees that formula (9) makes sense for all i and k in the range, and none of the determinants involved is identically zero.

Now, denote the upper left determinant in the right hand side by Δ_{i+1jk} . Then, clearly, the remaining three are Δ_{i-1jk} , Δ_{ijk+1} and Δ_{ijk-1} ,

$$(10) \quad Y_{ijk} = \frac{\Delta_{i+1jk} \Delta_{i-1jk}}{\Delta_{ijk+1} \Delta_{ijk-1}},$$

and likewise

$$\Delta_{ij+1k} = |\vec{x}_a \dots \vec{x}_{b-1} \vec{x}_c \dots \vec{x}_{d-1}| \quad \Delta_{ij-1k} = |\vec{x}_{a+1} \dots \vec{x}_b \vec{x}_{c+1} \dots \vec{x}_d|,$$

Hence, by eqs. (7),

$$1 + Y_{ijk} = \frac{\Delta_{ij+1k} \Delta_{ij-1k}}{\Delta_{ijk+1} \Delta_{ijk-1}} \quad \frac{1}{1 + 1/Y_{ijk}} = \frac{\Delta_{i+1jk} \Delta_{i-1jk}}{\Delta_{ij+1k} \Delta_{ij-1k}},$$

and therefore, indeed,

$$\begin{aligned} Y_{ij+1k} Y_{ij-1k} &= \frac{\Delta_{i+1j+1k} \Delta_{i-1j+1k}}{\Delta_{ij+1k+1} \Delta_{ij+1k-1}} \frac{\Delta_{i+1j-1k} \Delta_{i-1j-1k}}{\Delta_{ij-1k+1} \Delta_{ij-1k-1}} \\ &= \frac{\Delta_{i+1j+1k} \Delta_{i+1j-1k}}{\Delta_{i+1jk+1} \Delta_{i+1jk-1}} \frac{\Delta_{i-1j+1k} \Delta_{i-1j-1k}}{\Delta_{i-1jk+1} \Delta_{i-1jk-1}} \\ &\times \frac{\Delta_{i+1jk+1} \Delta_{i-1jk+1}}{\Delta_{ij+1k+1} \Delta_{ij-1k+1}} \frac{\Delta_{i+1jk-1} \Delta_{i-1jk-1}}{\Delta_{ij+1k-1} \Delta_{ij-1k-1}} = \frac{(1 + Y_{i+1jk})(1 + Y_{i-1jk})}{(1 + 1/Y_{ijk+1})(1 + 1/Y_{ijk-1})}, \end{aligned}$$

at least for $1 < i < r$ and $1 < k < r'$. The boundary cases are checked separately using the fact that, by construction,

$$\begin{aligned} \Delta_{0jk} &= \Delta_{0j-1k-1} & \Delta_{r+1jk} &= \Delta_{r+1j-1k+1} \\ \Delta_{ij0} &= \Delta_{i+1j-10} & \Delta_{ijr'+1} &= \Delta_{i-1j-1r'+1}. \end{aligned}$$

Thus, for instance,

$$\begin{aligned} Y_{1j+1k} Y_{1j-1k} &= \frac{\Delta_{2j+1k} \Delta_{0j+1k}}{\Delta_{1j+1k+1} \Delta_{1j+1k-1}} \frac{\Delta_{2j-1k} \Delta_{0j-1k}}{\Delta_{1j-1k+1} \Delta_{1j-1k-1}} = \frac{\Delta_{2j+1k} \Delta_{2j-1k}}{\Delta_{2jk+1} \Delta_{2jk-1}} \\ &\times \frac{\Delta_{2jk+1} \Delta_{0jk+1}}{\Delta_{1j+1k+1} \Delta_{1j-1k+1}} \frac{\Delta_{2jk-1} \Delta_{0jk-1}}{\Delta_{1j+1k-1} \Delta_{1j-1k-1}} = \frac{1 + Y_{2jk}}{(1 + 1/Y_{1jk+1})(1 + 1/Y_{1jk-1})} \end{aligned}$$

and

$$\begin{aligned} Y_{1j+11} Y_{1j-11} &= \frac{\Delta_{2j+11} \Delta_{0j+11}}{\Delta_{1j+12} \Delta_{1j+10}} \frac{\Delta_{2j-11} \Delta_{0j-11}}{\Delta_{1j-12} \Delta_{1j-10}} \\ &= \frac{\Delta_{2j+11} \Delta_{2j-11}}{\Delta_{2j2} \Delta_{2j0}} \frac{\Delta_{2j2} \Delta_{0j2}}{\Delta_{1j+12} \Delta_{1j-12}} = \frac{1 + Y_{2j1}}{1 + 1/Y_{1j2}}, \end{aligned}$$

as it should be (see Introduction). The remaining six boundary cases are checked similarly, and this completes this part of the proof.

A remark is in order here. In our approach determinants Δ_{ijk} play a secondary part, but, as it turns out, they satisfy their own system of algebraic equations rather similar to the Y-system. Indeed, note that either of the relations (7) is equivalent to the classical bilinear determinant identity

$$\begin{aligned} |\vec{x}_1 \vec{x}_3 \vec{x}_5 \dots \vec{x}_{r+3}| |\vec{x}_2 \vec{x}_4 \vec{x}_5 \dots \vec{x}_{r+3}| \\ = |\vec{x}_1 \vec{x}_2 \vec{x}_5 \dots \vec{x}_{r+3}| |\vec{x}_3 \vec{x}_4 \vec{x}_5 \dots \vec{x}_{r+3}| \\ + |\vec{x}_1 \vec{x}_4 \vec{x}_5 \dots \vec{x}_{r+3}| |\vec{x}_2 \vec{x}_3 \vec{x}_5 \dots \vec{x}_{r+3}|, \end{aligned}$$

also known as the Plücker relation. Hence, the Δ 's satisfy the lattice system

$$\Delta_{ij+1k} \Delta_{ij-1k} = \Delta_{i+1jk} \Delta_{i-1jk} + \Delta_{ijk+1} \Delta_{ijk-1},$$

which is well known in Lattice Soliton Theory as the 3D Hirota equation. Since the Y-systems and this Hirota equation are related to each other by a simple change of variables (10), Zamolodchikov's conjecture could be easily reformulated in terms of the latter. We will not go into it here, though.

3. P , (b)

Here we check that our solution is indeed a general one. We begin with numerology.

It is easy to figure out that any solution to the Y-system is completely determined by its values at $j = -1$ and $j = 0$. There are $2rr'$ of those in the original formulation (1), but since we have eliminated half of the lattice, we only have half that number left, that is rr' . On the other hand, the solution we are looking at depends on $r + r' + 2$ points in \mathbb{CP}^r , but due to projective invariance, $r + 2$ of them can be chosen arbitrarily. This leaves r' points in an r -dimensional space, which also gives rr' . So the numbers match up, but now we have to prove that this is no accident.

First, a lemma.

Lemma. *Two solutions to the Y-system (2) coincide everywhere on the lattice $L = \{(i, j, k) \in \mathbb{Z}^3 \mid 1 \leq i \leq r, 1 \leq k \leq r', i + j + k \text{ even}\}$ if they coincide on either of the following subsets of this lattice:*

- a) $J_{-1} \cup J_0$,
- b) Q_0 ,
- c) $K_1 \cap (Q_0 \cup Q_1 \cup \dots \cup Q_{r'-1})$,

where J_n , K_n and Q_n are intersections of L with the planes $j = n$, $k = n$ and $i + j + k = -2n$ respectively.

The first item is rather obvious. It has already been used in the numerical part, and is of no further use here.

To prove item (b), consider first the equation of the Y-system (2) with $(i, j, k) = (1, -3, 1)$:

$$Y_{1-41} = \frac{1 + Y_{2-31}}{Y_{1-21} (1 + 1/Y_{1-32})}.$$

All three lattice points involved in the right hand side belong to Q_0 , while the point $(1, -4, 1)$ in the left hand side is the only point of the intersection $Q_1 \cap J_{-4}$. Hence, if two solutions coincide on Q_0 , they also coincide on $Q_1 \cap J_{-4}$. But now the next two equations,

$$Y_{2-51} = \frac{(1 + Y_{3-41})(1 + Y_{1-41})}{Y_{2-31} (1 + 1/Y_{2-42})}$$

$$Y_{1-52} = \frac{1 + Y_{2-42}}{Y_{1-32} (1 + 1/Y_{1-43}) (1 + 1/Y_{1-41})},$$

have the right hand sides entirely contained in $Q_0 \cup (Q_1 \cap J_{-4})$ and the left hand sides spanning $Q_1 \cap J_{-5}$. Hence, our solutions coincide on $Q_1 \cap J_{-5}$ as well.

Clearly, this process can be continued step by step to cover the entire Q_1 , and then repeated again and again for all Q_n with $n > 0$. That done, we can start over from $Q_{-1} \cap J_{2-r-r'}$ and move in the opposite direction to cover all Q_n with $n < 0$ as well. This settles item (b).

On to item (c), we begin with the $r(r' - 1)$ equations

$$1 + 1/Y_{ij2} = \frac{(1 + Y_{i+1j1})(1 + Y_{i-1j1})}{Y_{ij+11} Y_{ij-11}} \quad -2r' + 2 \leq i + j \leq -2.$$

These imply that if two solutions coincide on $K_1 \cap (Q_0 \cup Q_1 \cup \dots \cup Q_{r'-1})$, then they also coincide on $K_2 \cap (Q_0 \cup Q_1 \cup \dots \cup Q_{r'-2})$. Then the next $r(r' - 2)$ equations,

$$1 + 1/Y_{ij3} = \frac{(1 + Y_{i+1j2})(1 + Y_{i-1j2})}{Y_{ij+12} Y_{ij-12} (1 + 1/Y_{ij1})} \quad -2r' + 3 \leq i + j \leq -3,$$

give $K_3 \cap (Q_0 \cup Q_1 \cup \dots \cup Q_{r'-3})$, and so it continues all the way to $K_{r'} \cap Q_0$. But clearly, the union of these r' sets contains Q_0 , and item (c) has thus

reduced to item (b). This settles the lemma, and we return to the main proof.

Fix the points x_0, \dots, x_{r+1} in general position and write out formula (9) for $(i, j, k) \in K_1 \cap (Q_0 \cup Q_1 \cup \dots \cup Q_{r'-1})$ as r' systems of r equations each for r' unknown points $x_{r+2}, \dots, x_{r+r'+1}$:

$$\begin{aligned} \frac{|\vec{x}_0 \dots \vec{x}_i \vec{x}_{i+2} \dots \vec{x}_{r+1}| |\vec{x}_1 \dots \vec{x}_{i-1} \vec{x}_{i+1} \dots \vec{x}_{r+2}|}{|\vec{x}_0 \dots \vec{x}_{i-1} \vec{x}_{i+2} \dots \vec{x}_{r+2}| |\vec{x}_1 \dots \vec{x}_{r+1}|} &= Y_{i,-i-1,1} \\ \frac{|\vec{x}_1 \dots \vec{x}_{i+1} \vec{x}_{i+3} \dots \vec{x}_{r+2}| |\vec{x}_2 \dots \vec{x}_i \vec{x}_{i+2} \dots \vec{x}_{r+3}|}{|\vec{x}_1 \dots \vec{x}_i \vec{x}_{i+3} \dots \vec{x}_{r+3}| |\vec{x}_2 \dots \vec{x}_{r+2}|} &= Y_{i,-i-3,1} \\ &\vdots \\ \frac{|\vec{x}_{r'-1} \dots \vec{x}_{i+r'-1} \vec{x}_{i+r'+1} \dots \vec{x}_{r+r'}| |\vec{x}_{r'} \dots \vec{x}_{i+r'-2} \vec{x}_{i+r'} \dots \vec{x}_{r+r'+1}|}{|\vec{x}_{r'-1} \dots \vec{x}_{i+r'-2} \vec{x}_{i+r'+1} \dots \vec{x}_{r+r'+1}| |\vec{x}_{r'} \dots \vec{x}_{r+r'}|} &= Y_{i,-i-2r'+1,1} . \end{aligned}$$

Clearly, we shall be done if we show that these can be solved for any values of the Y 's in the right hand sides, except perhaps 0, -1 and ∞ .

Note that the first of the above systems contains only one unknown point, x_{r+2} , and its solution is easily found to be

$$x_{r+2} = t(x_0) ,$$

where t is the projective transformation whose linear counterpart T has $\vec{x}_1, \dots, \vec{x}_{r+1}$ for eigenvectors,

$$T \vec{x}_n = \lambda_n \vec{x}_n \quad n = 1, \dots, r+1 ,$$

and the respective eigenvalues are given by

$$\lambda_n = \prod_{i=n}^r \frac{1}{1 + \frac{1}{Y_{i,-i-1,1}}} .$$

It is also easy to show that the newfound point x_{r+2} is in general position relative to the points x_1, \dots, x_{r+1} . This lets us use the second system in the same way to find x_{r+3} , then use the third one to find x_{r+4} and so on all the way to the last point $x_{r+r'+1}$. Thus, the solution in question is indeed a general one, and the Theorem is fully proved.

As a final remark, note that the change of indices $(i, j, k) \rightarrow (r+1-i, j+r+r'+2, r'+1-k)$ translates, in terms of the numbers $a \dots d$, into the pairwise shift/permutation $(a, b, c, d) \rightarrow (c-r-r'-2, d-r-r'-2, a, b)$. But clearly, the latter has no effect on the general solution (5), and so our theorem can be improved somewhat by replacing the periodicity condition (4) with the stronger one:

$$Y_{r+1-i, j+r+r'+2, r'+1-k} = Y_{ijk} .$$

In fact, Zamolodchikov's conjecture has always been formulated in this stronger form. The only reason to start with the weaker version was to slightly simplify exposition.

R

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